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ON AN OPERATOR THEORY OF LINEAR SYSTEMS WITH PURE AND DISTRIBUTED DELAYS*

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Abstract

A representation theory based on convolution operations is developed for a large class of linear systems containing pure and distributed delays in state and control. In terms of this framework a necessary and sufficient condition and a sufficient condition are given for functional (null) controllability. The conditions involve the generation of modules defined over a convolution ring of functions.

1. Introduction

In many control problems the systems under consideration contain pure and distributed time delays in state and control (examples are given by MANITIUS [1]). Such systems are usually referred to as hereditary systems since the rate of change of the present state depends on past values of the state and control or input.

In this paper we consider the class of linear systems given by a first-order functional differential equation of the form

$$\dot{x}(t) = \int_{-c}^{0} A(\theta)x(t+\theta)d\theta + F_0 x(t) + \sum_{i=1}^{r} F_i x(t-a_i) + \int_{-d}^{0} B(\theta)u(t+\theta)d\theta + G_0 u(t) + \sum_{i=1}^{s} G_i u(t-b_i) \tag{1}$$

where c,d and the a_i, b_i are positive real numbers, the F_i (G_i) are $n\times n$ ($n\times m$) matrices over the reals R, $A(\theta)$ (resp. $B(\theta)$) is a $n\times n$ ($n\times m$) matrix of (Lebesgue) measurable and integrable functions on $[-c,0]$ ($[-d,0]$), $x(t)\in R^n$ is the "instantaneous state," and $u(t)\in R^m$ is the input.

Systems of the form (1) have been studied using mainly functional-analytical methods applied to a state space setting defined in terms of the product space $R^n \times L^p(-h,0;R^n)$ where $h = \max\{c,a_i\}$. In particular, numerous results on controllability and optimal feedback control can be found in the work of DELFOUR-MITTER [2,3], DELFOUR [4], and DELFOUR-McCALLA-MITTER [5] (see also the survey by MANITIUS [6]).

In contrast to existing methods, our approach to the study of (1) is based on an algebraic setting defined in terms of convolution operators. More precisely, in the next section it is shown that (1) can be written in the form

$$\dot{x}(t) = (F*x)(t) + (G*u)(t) \tag{2}$$

where * denotes convolution and F and G are matrices whose elements belong to a convolution ring of functions and impulses (Dirac distributions). The convolution representation (2) is a special case of the time-domain operator framework developed by KAMEN [7].

In the latter part of the paper the representation (2) is applied to the problem of driving initial functions to the zero function in finite time (functional null controllability). New algebraic criteria for controllability are given in terms of modules defined over the convolution rings.

2. Representation by Convolution Operators

Let L^{loc} denote the space of all real-valued Lebesgue measurable functions f(t) that are locally integrable, i.e. $\int_K |f(t)|dt < \infty$ for any compact subset K of R. Let L_+^{loc} denote the subspace of L^{loc} consisting of all functions with support bounded on the left. It is easily verified that L_+^{loc} is a ring with pointwise addition and with convolution defined by

$$(g*f)(t) = \int_{-\infty}^{\infty} g(\theta)f(t-\theta)d\theta \ .$$

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Note that if the support of g is contained in $[0,a]$, $a > 0$, then

$$(g * f)(t) = \int_0^a g(\theta)f(t-\theta)d\theta$$

and by a change of variables, we have

$$(g * f)(t) = \int_{-a}^{0} \hat{g}(\theta)f(t+\theta)d\theta \tag{3}$$

where $\hat{g}(\theta) = g(-\theta)$.

Letting δ_a denote the unit impulse (Dirac distribution) concentrated at the point $\{a\}$, define

$$J = \left\{ f + \sum_{i=1}^{q} b_i \delta_{a_i} : f \in L_+^{loc},\ a_i, b_i \in R,\ q = \text{positive integer} \right\}. \tag{4}$$

The addition in the definition of J can be taken in a formal sense or it can be viewed as addition in some space of distributions. The set J is an overring of L_+^{loc} with the convolution operation given by

$$\left(f+\sum b_i\delta_{a_i}\right) * \left(g+\sum c_i\delta_{d_i}\right) = f*g+\sum\left(c_i f(t-d_i)+b_i g(t-a_i)\right) + \sum_j\left(\sum_i b_j c_i \delta_{a_j+d_i}\right). \tag{5}$$

Note that δ_0 is the identity element of the ring J.

Finally, let L_+^1 denote the space of all real-valued measurable functions f(t) defined a. e. on R with support bounded on the left, such that $\int_{-\infty}^{\infty}|f(t)|dt < \infty$. The space L_+^1 is a subring of L_+^{loc}.

Via the above constructions we can now characterize (1) *in terms* of convolution operations.

Theorem 1: With $u \in (L_+^1)^m$ and $x \in J^n$, the system equation (1) can be written in the form $\dot{x}(t) = (F * x)(t) + (G * u)(t)$ where F (G) is a $n \times n$ $(n \times m)$ matrix over $J_c = \{\alpha \in J : \text{supp}\ \alpha$ is compact and contained in $[0,\infty)\}$.

Proof: Applying (3) and (5) to (1), we have that

$$\dot{x} = \left(\hat{A} + \sum_{i=o}^{r} F_i\delta_{a_i}\right) * x + \left(\hat{B} + \sum_{i=o}^{s} G_i\delta_{b_i}\right) * u$$

where $\hat{A}(t) = A(-t)$ and $\hat{B}(t) = B(-t)$.
By definition of A,B and the F_i, G_i, the coefficient matrices of x and u are over J_c. ■

3. Initial Conditions

In this section we show how to "handle" nonzero initial conditions in solving the operator equation $\dot{x} = F * x + G * u$. This result will be utilized in the next section to study functional controllability.

Suppose that we are given

$$\dot{x}(t) = (F * x)(t) + (G * u)(t) \tag{6}$$

with F and G defined over J_c as in Theorem 1. Now let $a = \max\{\tau \in supp\ F\}$ and $b = \max\{\tau \in \text{supp}\ G\}$. Then to solve (6) for $t > 0$ *we need to specify* x(t) *on the interval* $[-a,0]$ and u(t) on the interval $[-b,0]$. We assume the following initial conditions

$$\left.\begin{aligned} x(0) &= x_o \in R^n \\ x(t) &= \phi(t),\ t \in [-a,0),\ \phi \in \left(L^1_{[-a,0]}\right)^n \\ u(t) &= u_o(t),\ t \in [-b,0],\ u_o \in \left(L^1_{[-b,0]}\right)^m \end{aligned}\right\} \tag{7}$$

where $L^1_{[c,d]} = \{f \in L^1_+ : \text{supp}\ f \subseteq [c,d]\}$.

Given $f \in L_+^1$, define

$$f|_{[c,d]}(t) = \begin{cases} f(t), & t \in [c,d] \\ 0, & \text{otherwise} \end{cases}$$

By the results of DELFOUR [4], with initial data (7) and $u|_{(o,\infty)} \in \left(L^1_{[o,\infty)}\right)^m$, the convolution equation (6) has a unique solution x(t) with $x|_{(o,\infty)} \in \left(L^{loc}_{[o,\infty)}\right)^n$.

Theorem 2: Given $u|_{(o,\infty)} \in \left(L^1_{[o,\infty)}\right)^m$ with initial data (7), the solution of (6) for $t > 0$ is equal to the solution for $t > 0$ of

$$\dot{x}(t) = (F * x)(t) + (G * u)(t) + v(t)$$

with initial data equal to zero, where

$$v = x_o\delta_o + (F * \phi + G * u_o)|_{(o,\infty)} \in J_c^n$$

Proof: Clearly $x_o\delta_o$ sets up the initial value $x_o \in R^n$. Since $x = \phi + x|_{(o,\infty)}$ and $u = u|_{(o,\infty)} + u_o$ with $x_o = 0$, we have that

$$F * x + G * u = F * x|_{(o,\infty)} + G * u|_{(o,\infty)} + F * \phi + G * u_o$$

Hence

$$(F * x + G * u)|_{(o,\infty)} = F * x|_{(o,\infty)} + G * u|_{(o,\infty)} + (F * \phi + G * u_o)|_{(o,\infty)}. ■$$

As a consequence of Theorem 2, operational methods, such as those developed in [7], can be used to solve operational differential equations of the form (6) with nonzero initial conditions. We shall now apply this result to the study of controllability.

4. Controllability

Let L_c^1 denote the set of all $f \in L_+^1$ such that supp f is compact and contained in $[o,\infty)$. With the induced operations L_c^1 is a convolution ring contained in the ring J_c.

Definition: The system given by (6) is said to be (null) controllable if for any initial condition (7), there exists a control $u \in (L_c^1)^m$ such that the solution of (6) is zero for all $t > h$, some $h > 0$.

In terms of the following constructions we derive a necessary and sufficient condition for controllability.

Let p^n denote the n^{th} derivative of δ_o in the sense of distributions. The element p^n belongs to $\mathcal{D}'_+$, the convolution ring of Schwartz distributions on R with support bounded on the left (see [7]). Given $f \in L_+^{loc}$, *the derivative of f in the sense of distributions is equal to* $p * f$ (with the convolution $p * f$ carried out in $\mathcal{D}'_+$). Therefore (6) can be expressed completely in terms of convolution operations:

$$(pI - F) * x = G * u \tag{8}$$

where I is the $n \times n$ identity matrix.

Now let $J_c[p]$ denote the set of all finite sums $\sum \alpha_i * p^i$ where $\alpha_i \in J_c$. With the standard operations, $J_c[p]$ is a convolution ring of distributions (contained in $\mathcal{D}'_+$). The rings J_c and L_c^1 $(\subset J_c)$ are subrings of $J_c[p]$.

Letting $J_c[p]^n$ denote the space of n-element column vectors over $J_c[p]$, we have that $J_c[p]^n$ *is a free finite module over* $J_c[p]$ with componentwise addition and with multiplication defined by

$$\pi(\beta_1, \beta_2, \ldots, \beta_n)^{TR} = (\pi * \beta_1, \pi * \beta_2, \ldots, \pi * \beta_n)^{TR}$$

where π, $\beta_i \in J_c[p]$, and TR denotes the transpose.

Since L_c^1 is a subring of $J_c[p]$, by restricting the multiplication operation to L_c^1 we have that $J_c[p]^n$ is also a (nonfinite) L_c^1-module. Given $\gamma_1, \gamma_2, \ldots, \gamma_q \in J_c[p]^n$, let $\langle \gamma_1, \gamma_2, \ldots, \gamma_q \rangle_{L_c^1}$ denote the L_c^1-submodule of $J_c[p]^n$ generated over L_c^1 by $\gamma_1, \gamma_2, \ldots, \gamma_q$. That is, $\langle \gamma_1, \ldots, \gamma_q \rangle_{L_c^1}$ is the set of all sums $\sum \pi_i \gamma_i$, $\pi_i \in L_c^1$.

Theorem 3: The system given by (6) is controllable if and only if J_c^n is contained in the L_c^1-module generated over L_c^1 by the columns of the $n \times n$ *matrix* $(pI - F)$ *and the* $n \times m$ *matrix* G; *that* is,

$$J_c^n \subset \langle (pI - F), G \rangle_{L_c^1} \tag{9}$$

Proof: Suppose that (9) holds and the initial condition (7) is given. Then there exist $\alpha \in (L_c^1)^m$ and $\beta \in (L_c^1)^n$ such that

$$(pI - F) * \beta - G * \alpha = x_o \delta_o + (F * \phi + G * u_o)|_{(o,\infty)} \in J_c^n.$$

Therefore by (8) and Theorem 2, β is the solution of (6) with the given initial data and with input $u|_{(o,\infty)} = \alpha$. Hence the system is controllable.

Conversely, suppose that the system is controllable.

Define $e_i = (0\ 0\ \ldots\ 1\ \ 0\ \ldots\ 0)^{TR} \in R^n$, $i = 1,2,\ldots n$.
i^{th} place

By Theorem 2, $e_i \delta_o$ sets up the initial value $x_o = e_i$. Then since the system is controllable, by (8) and Theorem 2 there exist $u_i \in (L_c^1)^m$ and $x_i \in (L_c^1)^n$ such that

$$(pI - F) * x_i - G * u_i = e_i \delta_o, \quad i = 1,2,\ldots,n.$$

But $e_1 \delta_o, e_2 \delta_o, \ldots, e_n \delta_o$ is a basis of J_c^n as a J_c-module, so that given $\gamma \in J_c^n$, there exist $\gamma_i \in J_c$ such that $\gamma = \sum \gamma_i (e_i \delta_o)$. Hence

$$\gamma = (pI - F) * (\sum \gamma_i * x_i) - G * (\sum \gamma_i * u_i). \tag{10}$$

Then since $\alpha * f \in L_c^1$ for any $\alpha \in J_c$, $f \in L_c^1$, it follows from (10) that the columns of $(pI - F)$ and G generate J_c^n over L_c^1. ■

Corollary: The system (6) is controllable if and only if

$$e_i \delta_o \in \langle (pI - F), G \rangle_{L_c^1}, \quad i = 1, 2, \ldots, n$$

where $e_i = (0\ 0\ \ldots\ 1\ \ 0\ \ldots\ 0) \in R^n$.
i^{th} place

In the finite-dimensional case where F and G are over $R\delta_o$, it is well known (KALMAN [8]) that the system given by (6) is controllable if and only if the rank of the $n \times nm$ matrix $(G, FG, \ldots, F^{n-1}G)$ is equal to n. If F and G are viewed as matrices over R (rather than $R\delta_o$), this is equivalent to requiring that

$$\langle G, FG, \ldots, F^{n-1}G \rangle_R = R^n . \tag{11}$$

Let N be a fixed subring of J_c with $\delta_o \in N$. In view of (11) it is reasonable to ask if the condition

$$\langle G, FG, \ldots, F^{n-1}G \rangle_N = N^n \tag{12}$$

is necessary and sufficient for controllability when F and G are over N. The answer is that (12) is sufficient but not necessary. The proof of sufficiency will be given in an expanded version of this paper. The simple example below shows that (12) is not necessary.

Example: Suppose that $\dot{x}(t) = x(t) + u(t-a)$, $a > 0$. In this case $F = \delta_o$ and $G = \delta_a$ are over the subring

$R[\delta_a] = \{\sum b_i \delta_{ia} : b_i \in R\} \subset J_c$.

Since the inverse of δ_a is δ_{-a}, $\langle \delta_a \rangle_N \neq N$ for $N = R[\delta_a]$ or $N = J_c$.

However the system is controllable since there exist $\alpha, \beta \in L^1_c$ such that $(p - \delta_o) * \alpha + \delta_a * \beta = \delta_o$.

For instance we could take

$$\alpha(t) = \begin{cases} e^t, & 0 \le t \le a \\ \dfrac{e^a}{b-a}(t-b), & a \le t \le b \\ 0, & \text{otherwise} \end{cases}$$

$$\beta(t) = \begin{cases} \dfrac{e^a}{b-a}(t+a-b-1), & 0 \le t \le b-a \\ 0, & \text{otherwise} \end{cases}$$

where b is any fixed real number greater than a. The function $-\beta$ given above is a control for the initial condition $x(0) = 1$.

5. Further Applications

In addition to function controllability, there exist many problem areas that can be investigated using the convolution representation (6). For example, the results of SONTAG [9] can be carried over to this framework giving a necessary and sufficient algebraic criterion for Euclidean reachability.

A particularly interesting topic is the study of feedback control systems with $u = K * x$, $K = m \times n$ matrix over a subring N of J_c. Practical problems include the development of algebraic procedures for the design of K's to achieve stabilization or pole allocation. Results on this are already available in the case that F and G are over a subring N that is a principal ideal domain (see MORSE [10] and SONTAG [9]).

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